Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **38**, 1-15; **http://www.math.u-szeged.hu/ejqtde/**

Multiple positive solutions for second order impulsive boundary value problems in Banach spaces*

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Abstract By means of the fixed point index theory of strict set contraction operators, we establish new existence theorems on multiple positive solutions to a boundary value problem for second-order impulsive integro-differential equations with integral boundary conditions in a Banach space. Moreover, an application is given to illustrate the main result.

Keywords Fixed point index; Impulsive differential equation; Positive solution; Measure of noncompactness.

1 Introduction.

Impulsive differential equations can be used to describe a lot of natural phenomena such as the dynamics of populations subject to abrupt changes (harvesting, diseases, etc.), which cannot be described using classical differential equations. That is why in recent years they have attracted much attention of investigators (cf., e.g., [2, 3, 7, 8, 9]). Meanwhile, the boundary value problem

*This work was supported partially by the NSF of China (10771202), the Research Fund for Shanghai Key Laboratory for Contemporary Applied Mathematics (08DZ2271900) and the Specialized Research Fund for the Doctoral Program of Higher Education of China (2007035805).

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with integral boundary conditions has been the subject of investigations along the line with impulsive differential equations because of their wide applicability in various fields (cf., e.g., [1, 2, 6, 10]).

In [3], D. Guo discussed the following second-order impulsive differential equations

$$\begin{cases} -x'' = f(t,x), \quad t \neq t_k\ ,\ k = 1, 2, \cdots, m, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, \cdots, m, \\ ax(0) - bx'(0) = \theta, \quad cx(1) + dx'(1) = \theta, \end{cases}$$

where $f \in C[J \times P, P]$, $J = [0,1]$, P is a cone in real Banach space E, θ denotes the zero element of E. $I_k \in C[P,P]$, $0 < t_1 < \cdots < t_k < \cdots < t_m < 1$. $a \geq 0,\ b \geq 0,\ c \geq 0,\ d \geq 0$ and $ac + ad + bc > 0$.

In [1], A. Boucherif investigated the existence of positive solutions to the following boundary value problem

$$\begin{cases} y''(t) = f(t, y(t)), \quad 0 < t < 1, \\ y(0) - ay'(0) = \int_0^1 g_0(s)y(s)ds, \\ y(1) - by'(1) = \int_0^1 g_1(s)y(s)ds, \end{cases}$$

where $f : [0,1] \times R \to R$ is continuous, $g_0,\ g_1 : [0,1] \to [0,+\infty)$ are continuous and positive, a and b are nonnegative real parameters.

In [2], M. Feng, B. Du and W. Ge studied the existence of multiple positive solutions for a class of second-order impulsive differential equations with p-Laplacian and integral boundary conditions

$$\begin{cases} -(\phi_p(u'(t)))' = f(t, u(t)), \quad t \neq t_k, t \in (0,1), \\ -\Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \cdots, n, \end{cases}$$

subject to the following boundary condition: $u'(0) = 0,\ u(1) = \int_0^1 g(t)u(t)dt$, where $\phi_p(s)$ is a p-Laplacian operator, $0 < t_1 < \cdots < t_k < \cdots < t_n < 1$, $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, $I_k \in C([0,+\infty), [0,+\infty))$.

In this paper, we are concerned with the existence of multiple positive solutions of the following second-order impulsive differential equations with integral boundary conditions in real Banach space E

$$\begin{cases} x'' = f(t, x, x', Tx, Sx), \quad t \in J\ ,\ t \neq t_k, \\ \Delta x|_{t=t_k} = -I_k(x(t_k), x'(t_k)), \quad k = 1, 2, \cdots, m, \\ \Delta x'|_{t=t_k} = \overline{I}_k(x(t_k), x'(t_k)), \quad k = 1, 2, \cdots, m, \\ x(0) - ax'(0) = \theta, \\ x(1) - bx'(1) = \int_0^1 g(s)x(s)ds, \end{cases} \tag{1.1}$$

where $a + 1 > b > 1$, $J = [0,1]$, $J' = J \backslash \{t_1, \cdots t_m\}$, $0 < t_1 < \cdots < t_k < \cdots < t_m < 1$, θ denotes the zero element of Banach space E, T and S are the linear operators defined as follows

$$(Tx)(t) = \int_0^t k(t,s)x(s)\mathrm{d}s, \quad (Sx)(t) = \int_0^1 h(t,s)x(s)\mathrm{d}s,$$

in which $k \in C[\mathbb{D}, R_+], h \in C[\mathbb{D}_0, R_+], \mathbb{D} = \{(t,s) \in J \times J : t \geq s\}, \mathbb{D}_0 = \{(t,s) \in J \times J : 0 \leq t, s \leq 1\}, R_+ = [0, +\infty), \Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$, i.e., $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$, $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. By means of the fixed point index theory of strict set contraction operators, we establish new existence theorems on multiple positive solutions to (1.1). Moreover, an application is given to illustrate the main result.

Let us first recall some basic information on cone (see more from [4, 5]). Let E be a real Banach space and P be a cone in E which defined a partial ordering in E by $x \leq y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. P is called solid if its interior $\overset{\circ}{P}$ is nonempty. If $x \leq y$ and $x \neq y$, we write $x < y$. If P is solid and $y - x \in \overset{\circ}{P}$, we write $x \ll y$.

Let $PC[J, E]$=$\{x : x$ is a map from J into E such that $x(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$ and $x(t_k^+)$ exists for $k = 1, 2, 3, \cdots, m\}$ and

$$PC^1[J, E] := \{x \in PC[J, E] : \ x'(t) \text{ is continuous at } t \neq t_k,$$
$$\text{and } x'(t_k^+), x'(t_k^-) \text{ exist for } k = 1, 2, 3, \cdots, m\}.$$

Clearly, $PC[J, E]$ is a Banach space with the norm $\|x\|_{PC} = \sup\limits_{t \in J} \|x(t)\|$ and $PC^1[J, E]$ is a Banach space with the norm $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$.

By a positive solution of BVP (1.1), we mean a map $x \in PC^1[J, E] \cap C^2[J', E]$ such that $x(t) \geq \theta$, $x'(t) \geq \theta$, $x(t) \not\equiv \theta$ for $t \in J$ and $x(t)$ satisfies (1.1).

Let α, α_{PC^1} be the Kuratowski measure of noncompactness in E and $PC^1[J, E]$, respectively (see [4, 5], for further understanding). Moreover, we set $J_1 = [0, t_1]$, $J_k = (t_{k-1}, t_k]$ $(k = 2, 3, \cdots, m)$, and for $u_i \in P$, $i = 1, 2, 3, 4$,

$$f^\infty = \limsup_{\sum_{i=1}^4 \|u_i\| \to \infty} \max_{t \in J} \frac{\|f(t, u_1, u_2, u_3, u_4)\|}{\sum_{i=1}^4 \|u_i\|}, \qquad f^0 = \limsup_{\sum_{i=1}^4 \|u_i\| \to 0} \max_{t \in J} \frac{\|f(t, u_1, u_2, u_3, u_4)\|}{\sum_{i=1}^4 \|u_i\|},$$
$$I^\infty(k) = \limsup_{\|u_1\| + \|u_2\| \to \infty} \frac{\|I_k(u_1, u_2)\|}{\|u_1\| + \|u_2\|}, \qquad I^0(k) = \limsup_{\|u_1\| + \|u_2\| \to 0} \frac{\|I_k(u_1, u_2)\|}{\|u_1\| + \|u_2\|}.$$

Similarly, we denote $\overline{I}^\infty(k), \overline{I}^0(k)$.

The following lemmas are basic, which can be found in [5].

Lemma 1.1 *If $W \subset PC^1[J, E]$ is bounded and the elements of W' are equicontinuous on each J_k $(k = 1, 2, \cdots, m)$. Then $\alpha_{PC^1}(W) = \max\Big\{\sup_{t \in J} \alpha(W(t)), \sup_{t \in J} \alpha(W'(t))\Big\}$.*

Lemma 1.2 *Let K be a cone in real Banach space E and Ω be a nonempty bounded open convex subset of K. Suppose that $A : \overline{\Omega} \to K$ is a strict set contraction and $A(\overline{\Omega}) \subset \Omega$, when $\overline{\Omega}$ denotes the closure of Ω in K. Then the fixed-point index $i(A, \Omega, K) = 1$.*

2 Main results

(H$_1$) $f \in C[J \times P \times P \times P \times P, P]$, and for any $r > 0$, f is uniformly continuous on $J \times P_r^4$, $I_k, \overline{I}_k \in C[P \times P, P]$ $(k = 1, 2, \cdots, m)$ are bounded on $P_r \times P_r$, where $P_r = \{x \in P : \|x\| \leq r\}$.
(H$_2$) $g \in L^1[0, 1]$ is nonnegative, and $u \in [0, a + 1 - b)$, where $u = \int_0^1 (a + s)g(s)ds$.
(H$_3$) There exist nonnegative constants $c_i, d_k, \overline{d}_k,\ i = 1, 2, 3, 4,\ k = 1, 2$ such that

$$\alpha(f(t, B_1, B_2, B_3, B_4)) \leq \sum_{i=1}^{4} c_i \alpha(B_i), \forall\, t \in J\ ,\ B_i \subset P_r \ \ (i = 1, 2, 3, 4), \tag{2.1}$$

$$\alpha(I_k(B_1, B_2)) \leq d_1 \alpha(B_1) + d_2 \alpha(B_2),\ B_1, B_2 \subset P_r\ , \tag{2.2}$$

$$\alpha(\overline{I}_k(B_1, B_2)) \leq \overline{d}_1 \alpha(B_1) + \overline{d_2} \alpha(B_2),\ B_1, B_2 \subset P_r\ , \tag{2.3}$$

and

$$l = \max\{l_1, \quad l_2\} < 1,$$

where

$$l_1 = 2m_2(c_1 + c_2 + k^* c_3 + h^* c_4) + m_2 m(\overline{d}_1 + \overline{d}_2) + \overline{m}_2 m(d_1 + d_2),$$

$$l_2 = 2m_4(c_1 + c_2 + k^* c_3 + h^* c_4) + m_4 m(\overline{d}_1 + \overline{d}_2) + \overline{m}_4 m(d_1 + d_2),$$

in which

$$k^* = \max\{k(t, s), t, s \in \mathbb{D}\},\ h^* = \max\{h(t, s), t, s \in \mathbb{D}_0\}.$$

(H$_4$) $f^\infty = f^0 = 0$, $I^\infty(k) = I^0(k) = 0$, $\overline{I}^\infty(k) = \overline{I}^0(k) = 0$.

Lemma 2.1 *Let (H_1) and (H_2) hold. Then $x \in PC^1[J, E] \cap C^2[J', E]$ is a solution to (1.1) if and only if $x \in PC^1[J, E] \cap C^2[J', E]$ is a solution to the following impulsive integral equation:*

$$\begin{aligned} x(t) \quad &= \quad \int_0^1 H_1(t, s) f(s, x(s), x'(s), (Tx)(s), (Sx)(s)) \mathrm{d}s + \sum_{k=1}^{m} H_1(t, t_k) \overline{I}_k(x(t_k), x'(t_k)) \\ &\quad + \sum_{k=1}^{m} H_2(t, t_k) I_k(x(t_k), x'(t_k)), \end{aligned} \tag{2.4}$$

where

$$H_1(t, s) = G_1(t, s) + \frac{a + t}{a + 1 - b - u} \int_0^1 G_1(\tau, s) g(\tau) d\tau,$$

$$H_2(t, s) = G_2(t, s) + \frac{a + t}{a + 1 - b - u} \int_0^1 G_2(\tau, s) g(\tau) d\tau,$$

$$G_1(t,s)=\begin{cases}\frac{1}{a+1-b}(a+t)(b+s-1), & t\le s,\\ \frac{1}{a+1-b}(a+s)(b+t-1), & s\le t,\end{cases}$$

$$G_2(t,s)=\begin{cases}\frac{a+t}{a+1-b}, & t\le s,\\ \frac{b+t-1}{a+1-b}, & s\le t.\end{cases}$$

Proof. "$\Longrightarrow$".

Suppose that $x\in PC^1[J,E]\cap C^2[J',E]$ is a solution to problem (1.1). From (1.1), we get

$$x'(t)=x'(0)+\int_0^t f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s+\sum_{0<t_k<t}\overline{I}_k(x(t_k),x'(t_k)),$$

and

$$\begin{aligned}x(t) \quad = \quad & x(0)+tx'(0)+\int_0^t (t-s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s\\ & +\sum_{0<t_k<t}(t-t_k)\overline{I}_k(x(t_k),x'(t_k))-\sum_{0<t_k<t}I_k(x(t_k),x'(t_k)).\end{aligned} \tag{2.5}$$

In particular,

$$x'(1)=x'(0)+\int_0^1 f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s+\sum_{0<t_k<1}\overline{I}_k(x(t_k),x'(t_k)),$$

and

$$\begin{aligned}x(1) \quad = \quad & x(0)+x'(0)+\int_0^1 (1-s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s\\ & +\sum_{0<t_k<1}(1-t_k)\overline{I}_k(x(t_k),x'(t_k))-\sum_{0<t_k<1}I_k(x(t_k),x'(t_k)).\end{aligned}$$

From this and the boundary conditions in (1.1), and by induction, we obtain

$$x(0)=ax'(0),$$

and

$$\begin{aligned}x'(0) \quad = \quad & \frac{1}{a+1-b}\Big(\int_0^1 (b+s-1)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s\\ & +\sum_{0<t_k<1}(b+t_k-1)\overline{I}_k(x(t_k),x'(t_k))+\sum_{0<t_k<1}I_k(x(t_k),x'(t_k))\\ & +\int_0^1 g(s)x(s)ds\Big).\end{aligned}$$

This, together with (2.5), implies

$$
\begin{aligned}
x(t) &= \frac{a+t}{a+1-b}\Big(\int_0^1 (b+s-1)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s \\
&\quad+\sum_{0<t_k<1}(b+t_k-1)\overline{I}_k(x(t_k),x'(t_k))+\sum_{0<t_k<1}I_k(x(t_k),x'(t_k)) \\
&\quad+\int_0^1 g(s)x(s)ds\Big)+\int_0^t (t-s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s \\
&\quad+\sum_{0<t_k<t}(t-t_k)\overline{I}_k(x(t_k),x'(t_k))-\sum_{0<t_k<t}I_k(x(t_k),x'(t_k)) \\
&= \frac{1}{a+1-b}\int_0^t (a+s)(b+t-1)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s \\
&\quad+\frac{1}{a+1-b}\int_t^1 (a+t)(b+s-1)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s \\
&\quad+\frac{1}{a+1-b}\sum_{0<t_k<t}(a+t_k)(b+t-1)\overline{I}_k(x(t_k),x'(t_k)) \\
&\quad+\frac{1}{a+1-b}\sum_{t\le t_k<1}(a+t)(b+t_k-1)\overline{I}_k(x(t_k),x'(t_k)) \\
&\quad+\frac{1}{a+1-b}\sum_{0<t_k<t}(b+t-1)I_k(x(t_k),x'(t_k)) \\
&\quad+\frac{1}{a+1-b}\sum_{t\le t_k<1}(a+t)I_k(x(t_k),x'(t_k))+\frac{a+t}{a+1-b}\int_0^1 g(s)x(s)ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
x(t) &= \int_0^1 G_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s+\sum_{k=1}^m G_1(t,t_k)\overline{I}_k(x(t_k),x'(t_k)) \\
&\quad+\sum_{k=1}^m G_2(t,t_k)I_k(x(t_k),x'(t_k))+\frac{a+t}{a+1-b}\int_0^1 g(s)x(s)ds.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\int_0^1 g(t)x(t)dt &= \int_0^1 g(t)\Big(\int_0^1 G_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s \\
&\quad+\sum_{k=1}^m G_1(t,t_k)\overline{I}_k(x(t_k),x'(t_k)) \\
&\quad+\sum_{k=1}^m G_2(t,t_k)I_k(x(t_k),x'(t_k))+\frac{a+t}{a+1-b}\int_0^1 g(s)x(s)ds\Big)dt, \\
&= \int_0^1\int_0^1 g(t)G_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s\mathrm{d}t \\
&\quad+\int_0^1 g(t)\Big(\sum_{k=1}^m G_1(t,t_k)\overline{I}_k(x(t_k),x'(t_k))\Big)dt \\
&\quad+\int_0^1 g(t)\Big(\sum_{k=1}^m G_2(t,t_k)I_k(x(t_k),x'(t_k))\Big)dt+\int_0^1\frac{a+t}{a+1-b}g(t)dt\int_0^1 g(t)x(t)dt,
\end{aligned}
$$

and also,

$$
\begin{aligned}
\int_0^1 g(s)x(s)ds &= \frac{1}{1-\int_0^1 \frac{a+s}{a+1-b} g(s)ds}\Big(\int_0^1 (\int_0^1 G_1(\tau,s)g(\tau)d\tau) f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s \\
&\quad + \int_0^1 g(\tau)(\sum_{k=1}^m G_1(\tau,t_k)\overline{I}_k(x(t_k),x'(t_k)))d\tau \\
&\quad + \int_0^1 g(\tau)(\sum_{k=1}^m G_2(\tau,t_k)I_k(x(t_k),x'(t_k)))d\tau\Big).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
x(t) &= \int_0^1 G_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s \\
&\quad + \sum_{k=1}^m G_1(t,t_k)\overline{I}_k(x(t_k),x'(t_k)) + \sum_{k=1}^m G_2(t,t_k)I_k(x(t_k),x'(t_k)) \\
&\quad + \frac{a+t}{a+1-b-\int_0^1 (a+s)g(s)ds}\Big(\int_0^1 (\int_0^1 G_1(\tau,s)g(\tau)d\tau) f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s \\
&\quad + \int_0^1 g(\tau)(\sum_{k=1}^m G_1(\tau,t_k)\overline{I}_k(x(t_k),x'(t_k)))d\tau + \int_0^1 g(\tau)(\sum_{k=1}^m G_2(\tau,t_k)I_k(x(t_k),x'(t_k)))d\tau\Big) \\
&= \int_0^1 H_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s + \sum_{k=1}^m H_1(t,t_k)\overline{I}_k(x(t_k),x'(t_k)) \\
&\quad + \sum_{k=1}^m H_2(t,t_k)I_k(x(t_k),x'(t_k)).
\end{aligned}
$$

"$\Longleftarrow$"

If $x \in PC^1[J,E] \cap C^2[J',E]$ is a solution of Eq. (2.4), then a direct differentiation of (2.4) yields, for $t \neq t_k$

$$
\begin{aligned}
x'(t) &= \int_0^t \frac{a+s}{a+1-b} f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s \\
&\quad + \int_t^1 \frac{b+s-1}{a+1-b} f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s \\
&\quad + \sum_{0<t_k<t} \frac{a+t_k}{a+1-b}\overline{I}_k(x(t_k),x'(t_k)) + \sum_{t\leq t_k<1} \frac{b+t_k-1}{a+1-b}\overline{I}_k(x(t_k),x'(t_k)) \\
&\quad + \frac{1}{a+1-b}\sum_{k=1}^m I_k(x(t_k),x'(t_k)) + \frac{1}{a+1-b-\int_0^1 (a+s)g(s)ds} \\
&\quad \Big(\int_0^1 (\int_0^1 G_1(\tau,s)g(\tau)d\tau) f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s \\
&\quad + \int_0^1 g(\tau)(\sum_{k=1}^m G_1(\tau,t_k)\overline{I}_k(x(t_k),x'(t_k)))d\tau \\
&\quad + \int_0^1 g(\tau)(\sum_{k=1}^m G_2(\tau,t_k)I_k(x(t_k),x'(t_k)))d\tau\Big).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
x'(t) \;=\; & \int_0^1 H_1'(t,s) f(s, x(s), x'(s), (Tx)(s), (Sx)(s)) \mathrm{d}s + \\
& \sum_{k=1}^{m} H_1'(t,t_k)\overline{I}_k(x(t_k), x'(t_k)) + \sum_{k=1}^{m} H_2'(t,t_k) I_k(x(t_k), x'(t_k)),
\end{aligned} \tag{2.6}
$$

where

$$
H_1'(t,s) = G_1'(t,s) + \frac{1}{a+1-b-u}\int_0^1 G_1(\tau,s)g(\tau)d\tau,
$$

$$
H_2'(t,s) = \frac{1}{a+1-b} + \frac{1}{a+1-b-u}\int_0^1 G_2(\tau,s)g(\tau)d\tau,
$$

$$
G_1'(t,s) = \begin{cases} \frac{b+s-1}{a+1-b}, & t \le s, \\ \frac{a+s}{a+1-b}, & s \le t. \end{cases}
$$

Differentiating (2.6), we see

$$
x''(t) = f(t, x(t), x'(t), (Tx)(t), (Sx)(t)).
$$

Clearly,

$$
\Delta x|_{t=t_k} = -I_k(x(t_k), x'(t_k)), \quad \Delta x'|_{t=t_k} = \overline{I}_k(x(t_k), x'(t_k)),
$$

$$
x(0) - ax'(0) = \theta, \quad x(1) - bx'(1) = \int_0^1 g(s)x(s)ds.
$$

The proof is then complete.□

The following "Facts" are clearly known.

Fact I. For $t, s \in [0,1]$, we have

$$
\frac{a(b-1)}{a+1-b} \le G_1(t,s) \le \frac{(a+1)b}{a+1-b},
$$

$$
\frac{b-1}{a+1-b} \le G_2(t,s) \le \frac{a+1}{a+1-b},
$$

$$
\frac{b-1}{a+1-b} \le G_1'(t,s) \le \frac{a+1}{a+1-b}.
$$

Fact II. For $t, s \in [0,1]$, there exist positive constants m_i ,$\overline{m}_i$ $(i = 1,2,3,4)$ such that

$$
m_1 = \frac{a(b-1)}{a+1-b} + \frac{a^2(b-1)u_1}{u_2} \le H_1(t,s) \le \frac{(a+1)b}{a+1-b} + \frac{(a+1)^2 b u_1}{u_2} = m_2 \ ,
$$

$$
\overline{m}_1 = \frac{b-1}{a+1-b} + \frac{a(b-1)u_1}{u_2} \le H_2(t,s) \le \frac{a+1}{a+1-b} + \frac{(a+1)^2 u_1}{u_2} = \overline{m}_2 \ ,
$$

$$
m_3 = \frac{b-1}{a+1-b} + \frac{a(b-1)u_1}{u_2} \le H_1'(t,s) \le \frac{a+1}{a+1-b} + \frac{(a+1)b u_1}{u_2} = m_4 \ ,
$$

$$\overline{m}_3 = \frac{1}{a+1-b} + \frac{(b-1)u_1}{u_2} \leq H_2'(t,s) \leq \frac{1}{a+1-b} + \frac{(a+1)u_1}{u_2} = \overline{m}_4 \ ,$$

where

$$u_1 = \int_0^1 g(s)ds, \quad u_2 = (a+1-b-u)(a+1-b).$$

We shall reduce BVP (1.1) to an impulsive integral equation in E. To this end, we first consider operator A defined by

$$\begin{aligned}(Ax)(t) \quad &= \quad \int_0^1 H_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\mathrm{d}s+ \\ &\sum_{k=1}^m H_1(t,t_k)\overline{I}_k(x(t_k),x'(t_k)) + \sum_{k=1}^m H_2(t,t_k)I_k(x(t_k),x'(t_k)).\end{aligned} \tag{2.7}$$

In what follows, we write

$$Q = \{x \in PC^1[J,E] : x(t) \geq \theta, x'(t) \geq \theta, t \in J\}, \quad B_r = \{x \in PC^1[J,E] : \|x\|_{PC^1} \leq r\}.$$

Obviously, Q is a cone in space $PC^1[J,E]$.

Lemma 2.2 *Let* $(\mathrm{H}_1) - (\mathrm{H}_3)$ *hold. Then for any* $r > 0$, $A : Q \cap B_r \to Q$ *is a strict set contraction.*

Proof. By (H_1) and (H_2), we know that $A : Q \cap B_r \to Q$ is continuous and bounded. Let $C \subset Q \cap B_r$. From (2.6) and (2.7), it follows that the elements of $(AC)'$ are equicontinuous on each J_k $(k = 1, \cdots, m)$. Lemma 1.1 shows us that

$$\alpha_{PC^1}(AC) = \max\Big\{\sup_{t\in J}\alpha((AC)(t)), \sup_{t\in J}\alpha((AC)'(t))\Big\}.$$

By (2.7), we obtain

$$\begin{aligned}\alpha((AC)(t)) \quad &\leq \quad \alpha(\overline{co}\{H_1(t,s)f(s,x(s),x'(s),(Tx)(s),(Sx)(s)) : s \in [0,t], t \in J, \ x \in C\}) \\ &\quad + \sum_{k=1}^m \alpha(H_1(t,t_k)\overline{I}_k(C(t_k),C'(t_k))) + \sum_{k=1}^m \alpha(H_2(t,t_k)I_k(C(t_k),C'(t_k))) \\ &\leq \quad m_2\alpha(f(s,C(s),C'(s),(TC)(s),(SC)(s)), s \in J) \\ &\quad + m_2\sum_{k=1}^m \alpha(\overline{I}_k(C(t_k),C'(t_k))) + \overline{m}_2\sum_{k=1}^m \alpha(I_k(C(t_k),C'(t_k))) \\ &\leq \quad m_2\Big(c_1\alpha(C(J)) + c_2\alpha(C'(J)) + c_3\alpha((TC)(J)) + c_4\alpha((SC)(J))\Big) \\ &\quad + m_2\sum_{k=1}^m \Big(\overline{d}_1\alpha(C(t_k)) + \overline{d}_2\alpha(C'(t_k))\Big) + \overline{m}_2\sum_{k=1}^m \Big(d_1\alpha(C(t_k)) + d_2\alpha(C'(t_k))\Big).\end{aligned}$$

By

$$\alpha(C(J)) \leq 2\alpha_{PC^1}(C), \quad \alpha(C'(J)) \leq 2\alpha_{PC^1}(C), \tag{2.8}$$

$$\alpha(C(t_k)) \leq \alpha_{PC^1}(C), \quad \alpha(C'(t_k)) \leq \alpha_{PC^1}(C), \tag{2.9}$$

we have

$$\alpha((AC)(t)) \leq l_1 \alpha_{PC^1}(C).$$

In the same way, by virtue of (2.6)-(2.9) and (H_3), we get

$$\alpha((AC)'(t)) \leq l_2 \alpha_{PC^1}(C).$$

Thus,

$$\alpha_{PC^1}(AC) \leq l \alpha_{PC^1}(C).$$

Since $l < 1$, we assert that $A : Q \cap B_r \to Q$ is a strict set contraction.□

Theorem 2.1. *Let* $(\mathrm{H}_1) - (\mathrm{H}_4)$ *hold,* P *be normal and solid. Let there exist* $v \gg \theta$, $0 < t_* < t^* < 1$ *and* $\sigma \in C[I, R_+]$ $(I = [t_*, t^*])$ *such that* $I \subset J_k$ *for some* k, *and*

$$f(t, u_1, u_2, u_3, u_4) \geq \sigma(t) v \quad (\forall\ t \in I),$$

$$u_1 \geq v, \quad u_i \geq \theta\ (i = 2, 3, 4), \quad \overline{m} \int_{t_*}^{t^*} \sigma(s) ds > 1,$$

where $\overline{m} = \min\{m_1, m_3\}$. *Then (1.1) has at least two positive solutions* $x_1, x_2 \in Q \cap C^2[J', E]$ *satisfying* $x_1(t) \gg v$ *and* $x_1'(t) \gg v$ *for* $t \in I$.

Proof. By Lemma 2.2, $A : Q \cap B_r \to Q$ is a strict set contraction. Write

$$\epsilon = \frac{1}{6(2 + k^* + h^*) m^{(1)}}\ , \ \epsilon_1 = \frac{1}{12 m m^{(1)}}\ , \ \epsilon_2 = \frac{1}{12 m m^{(1)}}, \tag{2.10}$$

where

$$m^{(1)} = \max\{m_2, \overline{m}_2, m_4, \overline{m}_4\}.$$

By (H_1) and (H_4), we know that there exist $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$ such that

$$\|f(t, u_1, u_2, u_3, u_4)\| \leq \varepsilon \sum_{i=1}^{4} \|u_i\| + M_1, \quad \forall\ t \in J,\ u_i \in P, \tag{2.11}$$

$$\|I_k(u_1, u_2)\| \leq \epsilon_1(\|u_1\| + \|u_2\|) + M_2, \quad \forall\ \ u_1,\ u_2 \in P, \tag{2.12}$$

$$\|\overline{I}_k(u_1, u_2)\| \leq \epsilon_2(\|u_1\| + \|u_2\|) + M_3, \quad \forall\ \ u_1,\ u_2 \in P. \tag{2.13}$$

Now, in view of (2.7), (2.10)-(2.13), we get

$$
\begin{aligned}
\|(Ax)(t)\| &\leq m_2\int_0^1 \|f(s,x(s),x'(s),(Tx)(s),(Sx)(s))\|\mathrm{d}s\\
&\quad +m_2\sum_{k=1}^m \|\overline{I}_k(x(t_k),x'(t_k))\| + \overline{m}_2\sum_{k=1}^m \|I_k(x(t_k),x'(t_k))\|\\
&\leq m_2\int_0^1 \Big(\varepsilon(\|x(s)\|+\|x'(s)\|+\|(Tx)(s)\|+\|(Sx)(s)\|)+M_1\Big)ds\\
&\quad +m_2\sum_{k=1}^m \Big(\epsilon_2(\|x(t_k)\|+\|x'(t_k)\|)+M_3\Big)+\overline{m}_2\sum_{k=1}^m \Big(\epsilon_1(\|x(t_k)\|+\|x'(t_k)\|)+M_2\Big)\\
&\leq m_2\Big(\varepsilon(2+k^*+h^*)\|x\|_{PC^1}+M_1\Big)+m_2m\Big(2\epsilon_2\|x\|_{PC^1}+M_3\Big)\\
&\quad +\overline{m}_2m\Big(2\epsilon_1\|x\|_{PC^1}+M_2\Big)\\
&= \Big((2+k^*+h^*)m_2\varepsilon+2m_2m\varepsilon_2+2\overline{m}_2m\varepsilon_1\Big)\|x\|_{PC^1}\\
&\quad +m_2M_1+m_2mM_3+\overline{m}_2mM_2\\
&\leq \frac{1}{2}\|x\|_{PC^1}+\overline{M}_1\ ,
\end{aligned}
\tag{2.14}
$$

where

$$\overline{M}_1 = m_2M_1+m_2mM_3+\overline{m}_2mM_2.$$

Similarly, from (2.6), (2.7), (2.10)-(2.13), we have

$$\|(Ax)'(t)\| \leq \frac{1}{2}\|x\|_{PC^1}+\overline{M}_2\ , \tag{2.15}$$

where

$$\overline{M}_2 = m_4M_1+m_4mM_3+\overline{m}_4mM_2.$$

It follows from (2.14) and (2.15) that

$$\|Ax\|_{PC^1} \leq \frac{1}{2}\|x\|_{PC^1}+\overline{M}\ , \tag{2.16}$$

where

$$\overline{M} = \max\{\overline{M}_1,\overline{M}_2\}.$$

On the other hand, the condition (H_4) implies that there exist $\overline{l}_1>0$, $\overline{l}_2>0$ and $\overline{l}_3>0$ such that

$$\|f(t,u_1,u_2,u_3,u_4)\| \leq \varepsilon\sum_{i=1}^4\|u_i\|,\quad \forall\ t\in J,\ u_i\in P,\ \sum_{i=1}^4\|u_i\|\leq \overline{l}_1, \tag{2.17}$$

$$\|I_k(u_1,u_2)\| \leq \epsilon_1(\|u_1\|+\|u_2\|),\quad \forall\ \ u_1,\ u_2\in P,\ \|u_1\|+\|u_2\|\leq \overline{l}_2, \tag{2.18}$$

$$\|\overline{I}_k(u_1,u_2)\| \leq \epsilon_2(\|u_1\|+\|u_2\|),\quad \forall\ \ u_1,\ u_2\in P,\ \|u_1\|+\|u_2\|\leq \overline{l}_3, \tag{2.19}$$

where $\varepsilon, \varepsilon_1, \varepsilon_2$ defined by (2.10).

Let $r_1 = \min\{\bar{l}_1, \bar{l}_2, \bar{l}_3\}$. Then by (2.6),(2.7),(2.17)-(2.19), we deduce that for $x \in Q$, $\|x\|_{PC^1} \leq \frac{r_1}{2+k^*+h^*}$,

$$\|Ax\|_{PC^1} \leq \frac{1}{2}\|x\|_{PC^1}. \tag{2.20}$$

Fix $R > \max\{2\overline{M}, 4\|v\|\}$. Let $\cup_1 = \{x \in Q, \|x\|_{PC^1} < R\}$. By (2.16), we have

$$\|Ax\|_{PC^1} \leq \frac{1}{2}\|x\|_{PC^1} + \overline{M} < \frac{1}{2}\|x\|_{PC^1} + \frac{1}{2}R \leq R,\ \forall\, x \in \overline{\cup}_1,$$

which gives

$$A(\overline{\cup}_1) \subset \cup_1. \tag{2.21}$$

Choose $0 < r < \min\{\|v\|\ , \frac{r_1}{2+k^*+h^*}\}$, and let $\cup_2 = \{x \in Q, \|x\|_{PC^1} < r\}$. Then by (2.20), we get

$$\|Ax\|_{PC^1} \leq \frac{1}{2}\|x\|_{PC^1} < r,$$

which implies

$$A(\overline{\cup}_2) \subset \cup_2. \tag{2.22}$$

Let $\cup_3 = \{x \in Q : \|x\|_{PC^1} < R, x(t) \gg v, x'(t) \gg v,\ \forall\, t \in [t_*, t^*]\}$. Then it is easy to check that $\cup_3$ is open in Q. Set $w(t) = 2v + 2tv$. Then $w \in Q$ and $w(t) \gg v,\ w'(t) \gg v$, for $t \in [t_*, t^*]$. Hence $w \in \cup_3$, and so, $\cup_3 \neq \emptyset$. By (2.21), we know that $\|Ax\|_{PC^1} < R,\ \forall\, x \in \overline{\cup}_3$. On the other hand, for $x \in \overline{\cup}_3$, we have

$$\begin{aligned} (Ax)(t) &\geq \int_{t_*}^{t^*} H_1(t,s) f(s, x(s), x'(s), (Tx)(s), (Sx)(s))\mathrm{d}s \\ &\geq m_1 \int_{t_*}^{t^*} f(s, x(s), x'(s), (Tx)(s), (Sx)(s))\mathrm{d}s \\ &\geq m_1 \int_{t_*}^{t^*} \sigma(s) v \mathrm{d}s \gg v, \end{aligned} \tag{2.23}$$

$$\begin{aligned} (Ax)'(t) &\geq \int_{t_*}^{t^*} H_1'(t,s) f(s, x(s), x'(s), (Tx)(s), (Sx)(s))\mathrm{d}s \\ &\geq m_3 \int_{t_*}^{t^*} f(s, x(s), x'(s), (Tx)(s), (Sx)(s))\mathrm{d}s \\ &\geq m_3 \int_{t_*}^{t^*} \sigma(s) v \mathrm{d}s \gg v. \end{aligned} \tag{2.24}$$

Therefore,

$$A(\overline{\cup}_3) \subset \cup_3. \tag{2.25}$$

Since $\cup_1$,$\cup_2$,$\cup_3$ are nonempty bounded open convex sets of Q, by (2.21), (2.22), (2.25) and Lemma 1.2, we see

$$i(A, \cup_i, Q) = 1,\ i = 1, 2, 3. \tag{2.26}$$

Clearly,

$$\cup_2 \subset \cup_1,\ \cup_3 \subset \cup_1,\ \cup_2 \cap \cup_3 = \emptyset. \tag{2.27}$$

It follows from (2.26) and (2.27) that

$$i(A, \cup_1 \backslash (\overline{\cup}_2 \cup \overline{\cup}_3), Q) = i(A, \cup_1, Q) - i(A, \cup_2, Q) - i(A, \cup_3, Q) = -1. \tag{2.28}$$

Finally, (2.26) and (2.28) yield that A has two fixed point $x_1 \in \cup_3$ and $x_2 \in \cup_1 \backslash (\overline{\cup}_2 \cup \overline{\cup}_3)$. It is easy to see that

$$x_1(t) \gg v \quad x_1'(t) \gg v, \qquad \text{for every } t \in [t_*, t^*],$$

and $\|x_2\|_{PC^1} > r$. Hence $x_1(t) \not\equiv \theta$ and $x_2(t) \not\equiv \theta$. The proof is then complete.□

3 An Example

Example 3.1. Consider the following boundary value problem for scalar second-order impulsive integro-differential equation

$$\begin{cases} x''(t) = 32\Big(x(t) + 2x'(t) + 3\displaystyle\int_0^t e^{-s}x(s)\mathrm{d}s + 4\int_0^1 e^{-2s}x(s)\mathrm{d}s\Big)^2 \\ \Big(1 + x(t) + x'(t) + \displaystyle\int_0^t e^{-s}x(s)\mathrm{d}s + \int_0^1 e^{-2s}x(s)\mathrm{d}s\Big)^{-2}, t \in J, t \neq t_1, \\ \Delta x|_{t_1=\frac{1}{2}} = -\dfrac{1}{100}\dfrac{(x(\frac{1}{2}))^2 + (x'(\frac{1}{2}))^2}{1 + (x(\frac{1}{2}))^2 + (x'(\frac{1}{2}))^2}, \\ \Delta x'|_{t_1=\frac{1}{2}} = \dfrac{1}{200}\dfrac{(x(\frac{1}{2}))^2 + (x'(\frac{1}{2}))^2}{1 + \Big(x(\frac{1}{2}) + x'(\frac{1}{2})\Big)^2}, \\ x(0) - 3x'(0) = 0, \\ x(1) - 2x'(1) = \displaystyle\int_0^1 \frac{1}{10}x(s)ds. \end{cases} \tag{3.1}$$

Conclusion. Problem (3.1) has at least two positive solutions $x_1(t)$ and $x_2(t)$ such that $x_1(t) > 1$, $x_1'(t) > 1$ for $t \in [\frac{1}{4}, \frac{1}{2}]$.

Proof. Let $E = R^1$ and $P = R_+$. Then P is a normal and solid cone in E and problem (3.1) can be regarded as a BVP in the form of (1.1) in E. In this case,

$$k(t,s) = e^{-s}, \quad h(t,s) = e^{-2s}, \quad a = 3, \quad b = 2, \quad m = 1,$$

$$t_1 = \frac{1}{2}, \quad g(s) = \frac{1}{10}, \quad t_* = \frac{1}{4}, \quad t^* = \frac{1}{2}, \quad v = 1,$$

and

$$f(t, u_1, u_2, u_3, u_4) = 32\Big(\frac{u_1 + 2u_2 + 3u_3 + 4u_4}{1 + u_1 + u_2 + u_3 + u_4}\Big)^2, \forall\, t \in J,\ u_i \geq 0,\ i = 1,2,3,4, \tag{3.2}$$

$$I_1(u_1,u_2)=\frac{1}{100}\frac{u_1^2+u_2^2}{1+u_1^2+u_2^2}\ , \tag{3.3}$$

$$\overline{I}_1(u_1,u_2)=\frac{1}{200}\frac{u_1^2+u_2^2}{1+\Big(u_1+u_2\Big)^2}. \tag{3.4}$$

Clearly,

$$f\in C[J\times P\times P\times P\times P,P],$$

$$I_1\in C[P\times P,P],\quad \overline{I}_1\in C[P\times P,P];$$

for any $r>0$, f is bounded and uniformly continuous on $J\times P_r\times P_r\times P_r\times P_r$, I_1 and $\overline{I}_1$ are bounded on $P_r\times P_r$. So (H_1) is satisfied.

$$u=\int_0^1(a+s)g(s)ds=\int_0^1(3+s)\frac{1}{10}ds=\frac{7}{20},\quad u\in[0,a+1-b)=[0,2).$$

This means that (H_2) is satisfied.

As in Example 3.2.1 in [5], we can prove that (2.1) is satisfied for $c_i=0$ $(i=1,2,3,4)$. By (3.3) and (3.4), we know that (2.2) and (2.3) are satisfied for

$$d_1=d_2=\frac{1}{50},\quad \overline{d}_1=\overline{d}_2=\frac{1}{100}.$$

By "Fact II", we have

$$m_1=\frac{39}{22},\quad m_2=\frac{164}{33},\quad \overline{m}_2=\frac{82}{33},\quad m_3=\frac{13}{22},m_4=\frac{74}{33},\quad \overline{m}_4=\frac{41}{66}.$$

So

$$l_1<\frac{11}{50},l_2<\frac{1}{10}$$

and $l<1$. Hence, (H_3) is satisfied.

Moreover, (3.2)-(3.4) implies that (H_4) holds.

On the other hand,

$$f(t,u_1,u_2,u_3,u_4)\geq 32\Big(\frac{u_1+u_2+u_3+u_4}{1+u_1+u_2+u_3+u_4}\Big)^2\geq 32\times\frac{1}{4}=8=\sigma(t),$$

$$\overline{m}=\min\{m_1,m_3\}=\frac{13}{22},\quad \overline{m}\int_{t_*}^{t^*}\sigma(s)ds=\frac{13}{22}\times\frac{1}{4}\times 8>1.$$

Thus, our conclusion follows from Theorem 2.1.□

4 Acknowledgements

The authors are grateful to the referees for very valuable comments and suggestions.

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(Received December 29, 2009)